Core 4 - June 2007

① a) $2x+1 \rightarrow$ sub in $x = -1/2$

$$f(-1/2) = 2(-1/2)^2 + (-1/2) - 3 = -3$$

b) $$\frac{(2x+3)(x-1)}{(x+1)(x-1)} = \frac{2x+3}{x+1}$$

② a) i) $$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}(x)^2 + \frac{(-1)(-2)(-3)}{3!}(x)^3$$

$$= 1 - x + x^2 - x^3$$

ii) $$\frac{1}{1+3x} = (1+3x)^{-1} = 1 - (3x) + (3x)^2 - (3x)^3$$

$$= 1 - 3x + 9x^2 - 27x^3$$

b) $$\frac{1+4x}{(1+x)(1+3x)} = \frac{A}{1+x} + \frac{B}{1+3x}$$

$$1 + 4x = A(1+3x) + B(1+x)$$

$\boxed{x = -1/3}$ $-1/3 = B(2/3) \rightarrow B = -1/2$

$\boxed{x = -1}$ $-3 = A(-2) \rightarrow A = 3/2$

$$= \frac{3}{2(1+x)} - \frac{1}{2(1+3x)}$$

c) i) $$\frac{3}{2(1+x)} = 3/2\,(1+x)^{-1}$$

$$= 3/2\,[1 - x + x^2 - x^3]$$

$$= 3/2 - 3/2\,x + 3/2\,x^2 - 3/2\,x^3$$

$$\frac{-1}{2(1+3x)} = -1/2\,(1+3x)^{-1}$$

$$= -1/2\,[1 - 3x + 9x^2 - 27x^3]$$

$$= -1/2 + 3/2\,x - 9/2\,x^3 + 27/2\,x^3$$

combined: $1 - 3x^2 + 12x^3$

ii) $|x| < 1$ and $|3x| < 1$

so take most limiting $\rightarrow |3x| < 1 \rightarrow |x| < 1/3$

(3) a) $4\cos(x) + 3\sin(x) = R\cos(x + \alpha)$

$= R[\cos(x)\cos(\alpha) + \sin(x)\sin(\alpha)]$

$R = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$4 = R\cos(\alpha)$
$3 = R\sin(\alpha)$ $\rightarrow$ $3/4 = \tan(\alpha)$

$\alpha = 36.869... = 36.9^\circ$

$\rightarrow$ $5\cos(x - 36.9^\circ)$

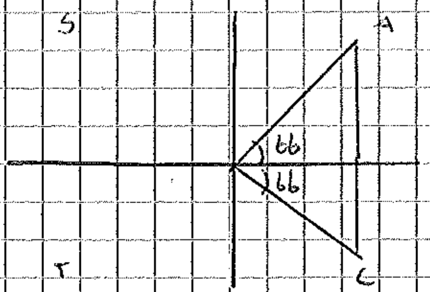
b) $5\cos(x - 36.9) = 2$

$\rightarrow \cos(x - 36.9) = 2/5$

$= \cos(t) = 0.4$ where $t = x - 36.9$

$t = 66.42..$ $0 < x < 360$

$-36.9 < t < 323.1$



$t = 66.42, 293.57..$

$\boxed{x = t + 36.9}$

$= 103.3^\circ, 330.4^\circ$ (10p)

c) $5\cos(x - 36.9)$ is stretch in y direction SF 5

$\rightarrow$ minimum value at -5

$\cos(x)$ has minimum value at $\cos^{-1}(-1) = 180^\circ$

$\therefore$ minimum value occurs at $180 + 36.9 = 216.9^\circ$

(4) a) i) $t = 0 \rightarrow x = 15 - 12e^0 = 15 - 12 = 3$ cm

ii) $t = 14 \rightarrow x = 15 - 12e^{-14/14} = 15 - 12e^{-1} = 10.6$ cm (3p)

b) i) $10 = 15 - 12e^{-t/14}$

$-5 = -12e^{-t/14}$

$5/12 = e^{-t/14}$

$\ln(5/12) = -t/14$

$14\ln(5/12) = -t$

$t = -14\ln(5/12) = 14\ln(5/12)^{-1} = 14\ln(12/5)$

ii) $t = 12.256.. = 12$ days

c) i) $x = 15 - 12e^{-t/14}$

$dx/dt = -1/14 \times -12e^{-t/14}$

if $x = 15 - 12e^{-t/14}$

$\Rightarrow x - 15 = -12e^{-t/14}$

$= -1/14\,(x - 15)$

$= 1/14\,(15 - x)$

ii) $x = 8 \Rightarrow 1/14\,(15-8) = 7/14 = 0.5$ cm per day

(5) a) $x = 1 \Rightarrow y + 4 = 5y^2$

$5y^2 - y - 4 = 0$

$(5y + 4)(y - 1) = 0$

$y = -4/5$ $\qquad$ $y = 1 \rightarrow a = 1$ as $a > 0$

b) $y + 4x = 5x^2y^2$

$dy/dx + 4 = 10xy^2 + 10x^2y\,dy/dx$

PRODUCT RULE:

$u = 5x^2 \qquad v = y^2$

$du/dx = 10x \qquad dv/dx = 2y\,dy/dx$

$\rightarrow 10xy^2 + 10x^2y\,dy/dx$

$x = 1,\ y = 1$

$\rightarrow dy/dx + 4 = 10 + 10\,dy/dx$

$dy/dx - 10\,dy/dx = 10 - 4$

$dy/dx\,(1 - 10) = 6$

$dy/dx = 6/-9 = -2/3$

c) $dy/dx = -2/3$

$x = 1,\ y = 1$ $\rightarrow$

$y - y_1 = m(x - x_1)$

$y - 1 = -2/3\,(x - 1)$

$y - 1 = -2/3\,x + 2/3$

$y = -2/3\,x + 5/3$

(6) a) i) $x = \cos(\theta)$ $\qquad$ $y = \sin(2\theta)$

$dx/d\theta = -\sin(\theta)$ $\qquad$ $dy/d\theta = 2\cos(2\theta)$

ii) $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx} = 2\cos(2\theta) \times \dfrac{1}{-\sin(\theta)}$

when $\theta = \pi/6$ $= 1 \times 1/-\tfrac{1}{2} = -2$

b) $y = \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

- $x = \cos(\theta)$
- $\sin^2(\theta) + \cos^2(\theta) = 1 \rightarrow \sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1 - x^2}$

$\therefore y = 2\sqrt{1 - x^2} \times x$

$y^2 = 4(1 - x^2)x^2$

$= 4x^2(1 - x^2) \qquad k = 4$

(7) a) use scalar product: $\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 3 - 6 + 3 = 0$

$a \cdot b = 0 \quad \therefore$ perpendicular

b) at intersection: (1) $8 + 3\lambda = -4 + \mu$

(2) $6 - 3\lambda = 2\mu$

(3) $-9 - \lambda = 11 - 3\mu$

use (1) & (2) (+)
$8 + 3\lambda = -4 + \mu$
$6 - 3\lambda = 2\mu$

$14 = -4 + 3\mu$
$18 = 3\mu \rightarrow \mu = 6$

use (2) $6 - 3\lambda = 12$
$-6 = 3\lambda \rightarrow \lambda = -2$

check in (3) $-9 - \lambda = 11 - 3\mu$
$-9 + 2 = 11 - 18$ ✓

$\therefore$ Intersection: $8 + 3\lambda = 8 + 3(-2) = 2$
$6 - 3\lambda = 6 - 3(-2) = 12$
$-9 - \lambda = -9 - (-2) = -7$
$= (2, 12, -7)$

~~(*) c) $\vec{AP} = \vec{AO} + \vec{OP} = \begin{pmatrix} 4 \\ 0 \\ -11 \end{pmatrix} + \begin{pmatrix} 2 \\ 12 \\ -7 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ -18 \end{pmatrix}$~~

~~$\vec{AB} = \vec{AP} + \vec{PB}$ $\qquad |AP| = \sqrt{6^2 + 12^2 + (-18)^2}$~~

~~$\therefore |AB| = |AP| + |PB|$ $\qquad = \sqrt{504} = 6\sqrt{14}$~~

~~$= 6\sqrt{14} + 6\sqrt{14}$ $\qquad = |PB|$~~

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(8) a) $\frac{dy}{dx} = \frac{\sqrt{1 + 2y}}{x^2}$

$$\frac{1}{\sqrt{1 + 2y}} \, dy = \frac{1}{x^2} \, dx$$

$$\int (1 + 2y)^{1/2} \, dy = \int x^{-2} \, dx$$

$$(1 + 2y)^{1/2} = -1/x + c$$

$$\sqrt{1 + 2y} = -1/x + c$$

$y = 4, x = 1$

$$\rightarrow \sqrt{9} = -1 + c \rightarrow c = 4$$

$$\therefore \sqrt{1 + 2y} = -1/x + 4$$

b) $1 + 2y = (-1/x + 4)^2$

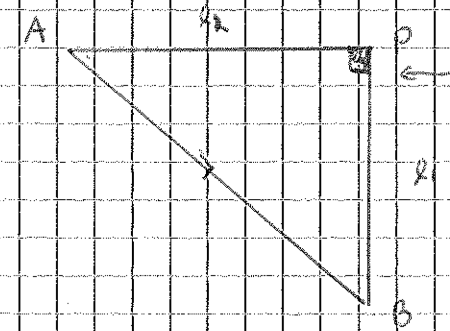
$$1 + 2y = 1/x^2 - 8/x + 16$$

$$2y = 1/x^2 - 8/x + 15$$

$$y = 1/2 \, (15 - 8/x + 1/x^2)$$

(9) a) $\vec{AP} = \vec{AO} + \vec{OP} = \begin{pmatrix} 4 \\ 0 \\ -11 \end{pmatrix} + \begin{pmatrix} 2 \\ 12 \\ -7 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ -18 \end{pmatrix}$

$$|AP| = \sqrt{6^2 + 12^2 + (-18)^2} = \sqrt{504} = |PB|$$



(perpendicular from part (a))

$$|AB| = \sqrt{(\sqrt{504})^2 + (\sqrt{504})^2}$$

$$= \sqrt{2 \times 504}$$

$$= 12\sqrt{7}$$